1 Пересчёт систем координат

1.1 Из декартовых в сферические

$$\rho = \sqrt{x^2 + y^2 + z^2} \tag{1}$$

$$\varphi = \arccos \frac{z}{\rho} \tag{2}$$

$$\theta = \operatorname{atan2}(y, x) \tag{3}$$

1.2 Из сферические в декартовы

$$x = \rho \cdot \sin \varphi \cdot \cos \theta \tag{4}$$

$$y = \rho \cdot \sin \varphi \cdot \sin \theta \tag{5}$$

$$z = \rho \cdot \cos \varphi \tag{6}$$

2 Базовые преобразования

2.1 Перенос точки

$$T^*(\Delta x, \Delta y, \Delta z) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{vmatrix} \tag{7}$$

2.2 Масштабирование точки относительно центра координат

$$S^*(Sx, Sy, Sz) = \begin{vmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{8}$$

2.3 Поворот точки вокруг оси x

$$R_x^*(\alpha) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{9}$$

2.4 Поворот точки вокруг оси y

$$R_y^*(\alpha) = \begin{vmatrix} \cos(\alpha) & 0 & -\sin(\alpha) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{10}$$

2.5 Поворот точки вокруг оси z

$$R_z^*(\alpha) = \begin{vmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{11}$$

3 Обратные операции

3.1 Перенос системы координат

$$T^{-1}(\Delta x, \Delta y, \Delta z) = T^{*}(-\Delta x, -\Delta y, -\Delta z) \tag{12}$$

3.2 Масштабирование осей системы координат

$$S^{-1}(Sx, Sy, Sz) = S^{*}\left(\frac{1}{Sx}, \frac{1}{Sy}, \frac{1}{Sz}\right) \tag{13}$$

3.3 Поворот системы координат вокруг осей

$$R_i^{-1}(\alpha) = R_i^{*}(-\alpha) \tag{14}$$

где: i – ось системы координат $(\{x, y, z\})$.

4 Композиция 3D преобразований

4.1 Поворот точки относительно линии, проходящей через начало системы координат на угол α

$$R_z^{-1}(\theta) \cdot R_y^{-1}(\varphi) \cdot \boldsymbol{R_z(\alpha)} \cdot R_y^{-1}(-\varphi) \cdot R_z^{-1}(-\theta) \tag{15}$$

$$\underbrace{\begin{vmatrix} \cos(-\theta) & \sin(-\theta) & 0 & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{R_z^{-1}(\theta)} \times \underbrace{\begin{vmatrix} \cos(-\varphi) & 0 & -\sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\varphi) & 0 & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{R_y^{-1}(\varphi)}$$

$$\times \underbrace{\begin{vmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{\boldsymbol{R_z(\alpha)}}$$

$$\times \underbrace{\begin{vmatrix} \cos(\varphi) & 0 & -\sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{R_y^{-1}(-\varphi)} \times \underbrace{\begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{R_z^{-1}(-\theta)}$$

$$\begin{aligned} R_z^{-1}(\theta) \cdot R_y^{-1}(\varphi) &= \begin{vmatrix} \cos(-\theta) & \sin(-\theta) & 0 & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \cos(-\varphi) & 0 & -\sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\varphi) & 0 & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \cos(\varphi) & 0 & \sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \cos(\theta)\cos(\varphi) & -\sin(\theta) & \cos(\theta)\sin(\varphi) & 0 \\ \sin(\theta)\cos(\varphi) & \cos(\theta) & \sin(\theta)\sin(\varphi) & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

WolframAlpha

$$\begin{aligned} R_y^{-1}(-\varphi) \cdot R_z^{-1}(-\theta) &= \begin{vmatrix} \cos(\varphi) & 0 & -\sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & -\sin(\varphi) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ \sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

WolframAlpha

$$\Big[R_z^{-1}(\theta)\cdot R_y^{-1}(\varphi)\Big]\cdot \boldsymbol{R_z(\alpha)} =$$

$$= \begin{vmatrix} \cos(\theta)\cos(\varphi) & -\sin(\theta) & \cos(\theta)\sin(\varphi) & 0 \\ \sin(\theta)\cos(\varphi) & \cos(\theta) & \sin(\theta)\sin(\varphi) & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\theta)\cos(\varphi)\cos(\alpha)+\sin(\theta)\sin(\alpha) & \cos(\theta)\cos(\varphi)\sin(\alpha)-\sin(\theta)\cos(\alpha) & \cos(\theta)\sin(\varphi) & 0 \\ \sin(\theta)\cos(\varphi)\cos(\alpha)-\cos(\theta)\sin(\alpha) & \sin(\theta)\cos(\varphi)\sin(\alpha)+\cos(\theta)\cos(\alpha) & \sin(\theta)\sin(\varphi) & 0 \\ -\sin(\varphi)\cos(\alpha) & -\sin(\varphi)\sin(\alpha) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

WolframAlpha

$$\Big[R_z^{-1}(\theta)\cdot R_y^{-1}(\varphi)\cdot \boldsymbol{R_z(\alpha)}\Big]\cdot \Big[R_y^{-1}(-\varphi)\cdot R_z^{-1}(-\theta)\Big] =$$

$$= \begin{vmatrix} \cos(\theta)\cos(\varphi)\cos(\alpha)+\sin(\theta)\sin(\alpha) & \cos(\theta)\cos(\varphi)\sin(\alpha)-\sin(\theta)\cos(\alpha) & \cos(\theta)\sin(\varphi) & 0 \\ \sin(\theta)\cos(\varphi)\cos(\alpha)-\cos(\theta)\sin(\alpha) & \sin(\theta)\cos(\varphi)\sin(\alpha)+\cos(\theta)\cos(\alpha) & \sin(\theta)\sin(\varphi) & 0 \\ -\sin(\varphi)\cos(\alpha) & -\sin(\varphi)\sin(\alpha) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\times \begin{vmatrix} \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & -\sin(\varphi) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ \sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{vmatrix}$$

$$\begin{aligned} r_{11} = {} & \big[\cos(\theta)\cos(\varphi)\cos(\alpha)+\sin(\theta)\sin(\alpha)\big]\cdot\big[\cos(\varphi)\cos(\theta)\big] \\ & + \big[\cos(\theta)\cos(\varphi)\sin(\alpha)-\sin(\theta)\cos(\alpha)\big]\cdot\big[-\sin(\theta)\big] \\ & + \big[\cos(\theta)\sin(\varphi)\big]\cdot\big[\sin(\varphi)\cos(\theta)\big] \end{aligned}$$

$$\begin{aligned} r_{12} = {} & \big[\cos(\theta)\cos(\varphi)\cos(\alpha)+\sin(\theta)\sin(\alpha)\big]\cdot\big[\cos(\varphi)\sin(\theta)\big] \\ & + \big[\cos(\theta)\cos(\varphi)\sin(\alpha)-\sin(\theta)\cos(\alpha)\big]\cdot\big[\cos(\theta)\big] \\ & + \big[\cos(\theta)\sin(\varphi)\big]\cdot\big[\sin(\varphi)\sin(\theta)\big] \end{aligned}$$

$$\begin{aligned} r_{13} = {} & \big[\cos(\theta)\cos(\varphi)\cos(\alpha)+\sin(\theta)\sin(\alpha)\big]\cdot\big[-\sin(\varphi)\big] \\ & + \big[\cos(\theta)\sin(\varphi)\big]\cdot\big[\cos(\varphi)\big] \end{aligned}$$

$$r_{14} = 0$$

$$
\begin{aligned}
r_{21} &= \big[\sin(\theta)\cos(\varphi)\cos(\alpha) - \cos(\theta)\sin(\alpha)\big] \cdot \big[\cos(\varphi)\cos(\theta)\big] \\
&\quad + \big[\sin(\theta)\cos(\varphi)\sin(\alpha) + \cos(\theta)\cos(\alpha)\big] \cdot \big[-\sin(\theta)\big] \\
&\quad + \big[\sin(\theta)\sin(\varphi)\big] \cdot \big[\sin(\varphi)\cos(\theta)\big]
\end{aligned}
$$

$$
\begin{aligned}
r_{22} &= \big[\sin(\theta)\cos(\varphi)\cos(\alpha) - \cos(\theta)\sin(\alpha)\big] \cdot \big[\cos(\varphi)\sin(\theta)\big] \\
&\quad + \big[\sin(\theta)\cos(\varphi)\sin(\alpha) + \cos(\theta)\cos(\alpha)\big] \cdot \big[\cos(\theta)\big] \\
&\quad + \big[\sin(\theta)\sin(\varphi)\big] \cdot \big[\sin(\varphi)\sin(\theta)\big]
\end{aligned}
$$

$$
\begin{aligned}
r_{23} &= \big[\sin(\theta)\cos(\varphi)\cos(\alpha) - \cos(\theta)\sin(\alpha)\big] \cdot \big[-\sin(\varphi)\big] \\
&\quad + \big[\sin(\theta)\sin(\varphi)\big] \cdot \big[\cos(\varphi)\big]
\end{aligned}
$$

$$
r_{24} = 0
$$

$$
\begin{aligned}
r_{31} &= \big[-\sin(\varphi)\cos(\alpha)\big] \cdot \big[\cos(\varphi)\cos(\theta)\big] \\
&\quad + \big[-\sin(\varphi)\sin(\alpha)\big] \cdot \big[-\sin(\theta)\big] \\
&\quad + \big[\cos(\varphi)\big] \cdot \big[\sin(\varphi)\cos(\theta)\big] \\
&= -\sin(\varphi)\cos(\alpha)\cos(\varphi)\cos(\theta) + \sin(\varphi)\sin(\alpha)\sin(\theta) + \cos(\varphi)\sin(\varphi)\cos(\theta) \\
&= \sin(\varphi)\left[\sin(\alpha)\sin(\theta) - \big(\cos(\alpha) - 1\big)\cos(\varphi)\cos(\theta)\right]
\end{aligned}
$$

$$
\begin{aligned}
r_{32} &= \big[-\sin(\varphi)\cos(\alpha)\big] \cdot \big[\cos(\varphi)\sin(\theta)\big] \\
&\quad + \big[-\sin(\varphi)\sin(\alpha)\big] \cdot \big[\cos(\theta)\big] \\
&\quad + \big[\cos(\varphi)\big] \cdot \big[\sin(\varphi)\sin(\theta)\big] \\
&= -\sin(\varphi)\cos(\alpha)\cos(\varphi)\sin(\theta) - \sin(\varphi)\sin(\alpha)\cos(\theta) + \cos(\varphi)\sin(\varphi)\sin(\theta) \\
&= -\sin(\varphi)\left[\big(\cos(\alpha) - 1\big)\cos(\varphi)\sin(\theta) + \sin(\alpha)\cos(\theta)\right]
\end{aligned}
$$

$$
\begin{aligned}
r_{33} &= \big[-\sin(\varphi)\cos(\alpha)\big] \cdot \big[-\sin(\varphi)\big] \\
&\quad + \big[\cos(\varphi)\big] \cdot \big[\cos(\varphi)\big] \\
&= \sin^2(\varphi)\cos(\alpha) + \cos^2(\varphi)
\end{aligned}
$$

$$
r_{34} = 0
$$

$$
\begin{aligned}
r_{41} &= 0 \\
r_{42} &= 0 \\
r_{43} &= 0 \\
r_{44} &= 1
\end{aligned}
$$